— Introduction to sub-Riemannian geometry —

[Agrachev, Barilari, Boscain – A comprehensive intro. to sub-Riem. geo, §3]

1. Sub-Riemannian structures
2. Admissible curves and minimal control
3. Distance

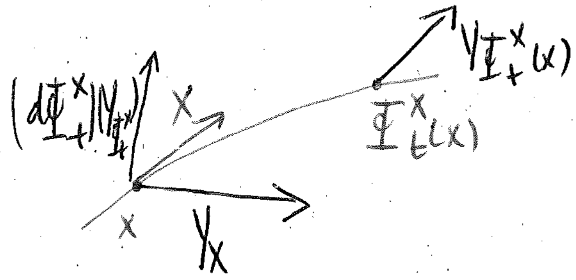
§ Sub-Riemannian structures

M: connected smooth mfld

$(\mathfrak{X}(M), [\cdot,\cdot])$: Lie algebra of vector fields on M

recall: $X, Y \in \mathfrak{X}(M)$,

$$[X,Y]_x := \lim_{t\to 0} \frac{(d\Phi^X_{-t})(Y_{\Phi^X_t(x)}) - Y_x}{t}, \quad x \in M$$



"fisherman's derivative"

$\mathcal{F} \subseteq \mathfrak{X}(M)$: sub-family

$\operatorname{Lie}\mathcal{F} :=$ smallest subalgebra of $(\mathfrak{X}(M), [\cdot,\cdot])$ containing $\mathcal{F}$

$\quad \doteq \operatorname{span}\{[X_1, \ldots, [X_{j-1}, X_j]] \mid X_i \in \mathcal{F}, j \in \mathbb{N}\}$

defn: $\mathcal{F}$ is bracket generating if

$$\mathrm{Lie}_q\mathcal{F} := \{X(q) \mid X \in \mathrm{Lie}\,\mathcal{F}\} \stackrel{!}{=} T_qM \ , \ \forall q \in M$$

note: $s(q)$ = minimal number of bracket to generate T_qM

$\rightsquigarrow$ $s(q)$ may vary on $M \ni q$, may even be unbounded

defn: $(M, \mathcal{U}, f)$ is a sub-Riemannian manifold if

$$\begin{tikzcd}
(\mathcal{U}_q, \langle\cdot,\cdot\rangle_q) \arrow[r, leftrightarrow] & \mathcal{U} \arrow[r, "f"] \arrow[d] & TM \arrow[dl] \\
& M &
\end{tikzcd}$$

smooth morphism of vector bundle

Smooth Euclidean vector bundle

and $\mathcal{D} := f(\Gamma(\mathcal{U})) \subseteq \mathfrak{X}(M) = \Gamma(TM)$

is bracket generating

$\leftarrow$ to be explained later why

$q \in M$

$\mathcal{D}_q := f(\mathcal{U}_q) \subseteq T_qM$: sub-v.s.

$= \{X(q) \mid X \in \mathcal{D}\}$

note: $m = \mathrm{rank}\,\mathcal{U} = \dim \mathcal{U}_q$, $n = \dim M$, $r(q) = \dim \mathcal{D}_q$

$\Rightarrow \quad r(q) \leq \min\{m, n\}$

§ Admissible curves

$(M, \mathcal{U}, f)$: sub-Riemannian mfld

$\gamma : [0,T] \longrightarrow M$: Lipschitz ($\Rightarrow$ differentiable a.e.)

<u>defn</u> : γ is <u>admissible</u> if $\exists\, u : [0,T] \longrightarrow \mathcal{U}$ <u>control</u> st

* u measurable , $u \in L^\infty([0,T], \mathcal{U})$

* $u(t) \in \mathcal{U}_{\gamma(t)}$

* $\dot\gamma(t) = f(\gamma(t), u(t))$, $\forall$ a.e. $t \in [0,T]$

<u>locally</u> : $q \in M$

$$f \downarrow \begin{array}{l} O_q \times \mathbb{R}^m : \text{trivialization of } \mathcal{U} \\ O_q \times \mathbb{R}^n : \text{trivialization of } TM \end{array}$$

$$\rightsquigarrow \quad f(q,u) = \sum_{i=1}^m u_i f_i(q)$$

$\gamma : [0,T] \to M$: $q = \gamma(0)$, $T \ll 1$, Lipschitz

γ admissible $\leftrightarrow$ $\exists\, u \in L^\infty([0,T], \mathbb{R}^m)$ st

$$\dot\gamma(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t))$$

$\overset{\text{ODE}}{\Longrightarrow}$ existence of control for small times & of admissible curves observation : γ : admissible

$\Rightarrow$ depending on f, may exist several control

observation
(more interesting!) $f(q,\cdot) : \mathcal{U}_q \longrightarrow T_qM$ linear!

$v \in T_qM$

$\Rightarrow \{ u \in \mathcal{U}_q \mid f(q,u) = v \} \subseteq \mathcal{U}_q$

is an affine subspace!

$\Rightarrow |\cdot|_q : \{ f(q,\cdot) = v \} \subseteq \mathcal{U}_q \longrightarrow \mathbb{R}_{\geq 0}$

if non-empty

has a unique minimum

(orthogonal projection)

$\gamma : [0,T] \to M$: Lipschitz ($\Rightarrow$ differentiable a.e.)

$u^*(t) := \operatorname{argmin} \{ |u| : u \in \mathcal{U}_q, \ f(q,u) = \dot{\gamma}(t) \}$

$\rightsquigarrow \ u^* : [0,T] \longrightarrow \mathcal{U}$ (defined ptwise a.e.)

lemma : γ admissible

$\Rightarrow u^*$ is a control of γ called minimal control

proof : we prove that u^* is measurable

step 1 : $t \longmapsto |u^*(t)|$ measurable

step 2 : $t \longmapsto u^*(t)$ measurable

<u>step 1</u>: $t \longmapsto |u^*(t)|$ measurable

$\overset{\text{defn}}{\iff} \quad A := \{t : |u^*(t)| \leq r\}$ measurable, $\forall r$

defn of $u^*(t)$ $= \{t : \exists u \in \cancel{\mathcal{U}_{\gamma(t)}}\ \mathbb{R}^m \text{ st } |u| \leq r,\ f(\gamma(t), u) = \dot{\gamma}(t)\}$

enough to work locally!

$\{u_i\}_{i \in \mathbb{N}} \subseteq \{|u| \leq r\} \subseteq \cancel{\mathcal{U}_{\gamma(t)}}\ \mathbb{R}^m$: dense

$$\Rightarrow \quad A = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \{t : \|f(\gamma(t), u_i) - \dot{\gamma}(t)\| < 1/j\}$$

measurable! any norm on $\cancel{T_{\gamma(t)}M}\ \mathbb{R}^n$

<u>step 2</u>: $t \longmapsto u^*(t)$ measurable

$\overset{\text{defn}}{\iff} \quad A := \{t : u^*(t) \in O\}, \quad \forall O \subseteq \mathcal{U}$: closed ball

uniqueness of solution $\overset{\downarrow}{=} \{t : \exists u \in O \text{ st } |u| = |u^*(t)|,\ f(\gamma(t), u) = \dot{\gamma}(t)\}$

$\{u_i\}_{i \in \mathbb{N}} \subseteq O$: dense

$$\Rightarrow \quad A = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} \{t : |u_i| < |u^*(t)| + 1/j,\ \|f(\gamma(t), u_i) - \dot{\gamma}(t)\| < 1/j\}$$

measurable!

$\square$

<u>defn</u>: $\gamma : [0, T] \longrightarrow M$: admissible

$$\text{length}(\gamma) := \int_0^T |u^*(t)|\, dt$$

eg: $M = \mathbb{R}^2$

$\mathcal{U} = \mathbb{R}^2 \times \mathbb{R}^2$

$f_1 : \mathcal{U} \to TM$

$(x, y, u_1, u_2) \mapsto (x, y, u_1, x u_2)$

$f_2 : \mathcal{U} \to TM$

$(x, y, u_1, u_2) \mapsto (x, y, u_1, x^2 u_2)$

$\gamma : [-1, 1] \to \mathbb{R}^2$

$t \mapsto (t, t^2)$

$\dot{\gamma}(t) = (1, 2t)$

$\Rightarrow (\gamma(t), \dot{\gamma}(t)) = f_1(t, t^2, 1, 2) = f_2(t, t^2, 1, 2/t)$

$\Rightarrow (\gamma(t), \dot{\gamma}(t)) \in f_1(\mathcal{U}_{\gamma(t)}), f_2(\mathcal{U}_{\gamma(t)}) \quad \forall$ a.e. t

1. γ is admissible in $(M, \mathcal{U}, f_1)$

$u(t) = (1, 2) \in L^\infty$

2. γ is <u>not</u> admissible in $(M, \mathcal{U}, f_2)$

$u(t) = (1, 2/t) \notin L^\infty$

3. minimal control of γ in $(M, \mathcal{U}, f_1)$

$$u^*(t) = \begin{cases} (1,2) & , t \neq 0 \\ (1,0) & , t = 0 \end{cases}$$ (not cts)